P.4

Complex Numbers

Our system of numbers developed as the need arose. Numbers were first used for counting. As society advanced, the rational numbers were formed to express fractional parts and ratios. Negative numbers were invented to express losses or debts. When it was discovered that the exact size of some very real objects could not be expressed with rational numbers, the irrational numbers were added to the system, forming the set of real numbers. Later still, there was a need for another expansion to the number system. In this section we study that expansion, the set of complex numbers.

Definitions

In Section P.3 we saw that there are no even roots of negative numbers in the set of real numbers. So, the real numbers are inadequate or incomplete in this regard. The imaginary numbers were invented to complete the set of real numbers. Using real and imaginary numbers, every nonzero real number has *two* square roots, *three* cube roots, *four* fourth roots, and so on. (Actually finding all of the roots of any real number is done in trigonometry.)

The imaginary numbers are based on the symbol $\sqrt{-1}$. Since there is no real number whose square is -1, a new number called i is defined such that $i^2 = -1$.

Definition: Imaginary Number i

The **imaginary number** i is defined by

$$i^2 = -1.$$

We may also write $i = \sqrt{-1}$.

A complex number is formed as a real number plus a real multiple of i.

Definition: Complex Numbers

The set of **complex numbers** is the set of all numbers of the form $a + bi$, where a and b are real numbers.

In $a + bi$, a is called the **real part** and b is called the **imaginary part.** Two complex numbers $a + bi$ and $c + di$ are **equal** if and only if their real parts are equal ($a = c$) and their imaginary parts are equal ($b = d$). If $b = 0$, then $a + bi$ is a **real number.** If $b \neq 0$, then $a + bi$ is an imaginary number.

The form $a + bi$ is the **standard form** of a complex number, but for convenience we use a few variations of that form. If either the real or imaginary part of a complex number is 0, then that part is omitted. For example,

$$0 + 3i = 3i, \qquad 2 + 0i = 2, \qquad \text{and} \qquad 0 + 0i = 0.$$

If b is a radical, then i is usually written before b. For example, we write $2 + i\sqrt{3}$ rather than $2 + \sqrt{3}i$, which could be confused with $2 + \sqrt{3i}$. If b is negative, a subtraction symbol can be used to separate the real and imaginary parts as in $3 + (-2)i = 3 - 2i$. A complex number with fractions, such as $\frac{1}{3} - \frac{2}{3}i$, may be written as $\frac{1 - 2i}{3}$.

Example 1 Standard form of a complex number

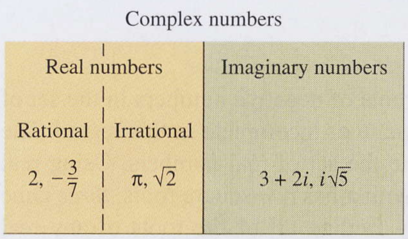
Determine whether each complex number is real or imaginary and write it in the standard form $a + bi$.

a. $3i$ **b.** 87 **c.** $4 - 5i$ **d.** 0 **e.** $\dfrac{1 + \pi i}{2}$

Solution

a. The complex number $3i$ is imaginary, and $3i = 0 + 3i$.
b. The complex number 87 is a real number, and $87 = 87 + 0i$.
c. The complex number $4 - 5i$ is imaginary, and $4 - 5i = 4 + (-5)i$.
d. The complex number 0 is real, and $0 = 0 + 0i$.
e. The complex number $\frac{1 + \pi i}{2}$ is imaginary, and $\frac{1 + \pi i}{2} = \frac{1}{2} + \frac{\pi}{2}i$.

Try This. Determine whether $i - 5$ is real or imaginary and write it in standard form. ■



■ **Figure P.23**

The real numbers can be classified as rational or irrational. The complex numbers can be classified as real or imaginary. The relationship between these sets of numbers is shown in Fig. P.23.

Addition, Subtraction, and Multiplication

Now that we have defined complex numbers, we define the operations of arithmetic with them.

Definition: Addition, Subtraction, and Multiplication

If $a + bi$ and $c + di$ are complex numbers, we define their sum, difference, and product as follows.

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$

$$(a + bi) - (c + di) = (a - c) + (b - d)i$$

$$(a + bi)(c + di) = (ac - bd) + (bc + ad)i$$

It is not necessary to memorize these definitions, because the results can be obtained by performing the operations as if the complex numbers were binomials with i being a variable, replacing i^2 with -1 wherever it occurs.

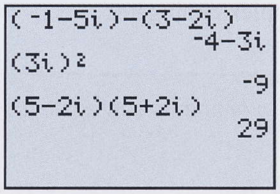
Example 2 Operations with complex numbers

Perform the indicated operations with the complex numbers.

a. $(-2 + 3i) + (-4 - 9i)$ **b.** $(-1 - 5i) - (3 - 2i)$ **c.** $2i(3 + i)$
d. $(3i)^2$ **e.** $(-3i)^2$ **f.** $(5 - 2i)(5 + 2i)$

Solution

a. $(-2 + 3i) + (-4 - 9i) = -6 - 6i$
b. $(-1 - 5i) - (3 - 2i) = -1 - 5i - 3 + 2i = -4 - 3i$
c. $2i(3 + i) = 6i + 2i^2 = 6i + 2(-1) = -2 + 6i$
d. $(3i)^2 = 3^2 i^2 = 9(-1) = -9$



■ **Figure P.24**

e. $(-3i)^2 = (-3)^2 i^2 = 9(-1) = -9$
f. $(5 - 2i)(5 + 2i) = 25 - 4i^2 = 25 - 4(-1) = 29$

Check these results with a calculator that handles complex numbers, as in Fig. P.24.

Try This. Find the product $(4 - 3i)(1 + 2i)$. ■

We can find whole-number powers of i by using the definition of multiplication. Since $i^1 = i$ and $i^2 = -1$, we have

$$i^3 = i^1 \cdot i^2 = i(-1) = -i \qquad \text{and} \qquad i^4 = i^1 \cdot i^3 = i(-i) = -i^2 = 1.$$

The first eight powers of i are listed here.

$$\begin{array}{llll} i^1 = i & i^2 = -1 & i^3 = -i & i^4 = 1 \\ i^5 = i & i^6 = -1 & i^7 = -i & i^8 = 1 \end{array}$$

This list could be continued in this pattern, but any other whole-number power of i can be obtained from knowing the first four powers. We can simplify a power of i by using the fact that $i^4 = 1$ and $(i^4)^n = 1$ for any integer n.

Example 3 Simplifying a power of i

Simplify.

a. i^{83} **b.** i^{-46}

Solution

a. Divide 83 by 4 and write $83 = 4 \cdot 20 + 3$. So

$$i^{83} = (i^4)^{20} \cdot i^3 = 1^{20} \cdot i^3 = 1 \cdot i^3 = -i.$$

b. Since $-46 = 4(-12) + 2$, we have

$$i^{-46} = (i^4)^{-12} \cdot i^2 = 1^{-12} \cdot i^2 = 1(-1) = -1.$$

Try This. Simplify i^{35}. ■

Division of Complex Numbers

The complex numbers $a + bi$ and $a - bi$ are called **complex conjugates** of each other.

Example 4 Complex conjugates

Find the product of the given complex number and its conjugate.

a. $3 - i$ **b.** $4 + 2i$ **c.** $-i$

Solution

a. The conjugate of $3 - i$ is $3 + i$, and $(3 - i)(3 + i) = 9 - i^2 = 10$.
b. The conjugate of $4 + 2i$ is $4 - 2i$, and $(4 + 2i)(4 - 2i) = 16 - 4i^2 = 20$.
c. The conjugate of $-i$ is i, and $-i \cdot i = -i^2 = 1$.

Try This. Find the product of $3 - 5i$ and its conjugate. ■

In general we have the following theorem about complex conjugates.

Theorem: Complex Conjugates

> If a and b are real numbers, then the product of $a + bi$ and its conjugate $a - bi$ is the real number $a^2 + b^2$. In symbols,
> $$(a + bi)(a - bi) = a^2 + b^2.$$

We use the theorem about complex conjugates to divide imaginary numbers, in a process that is similar to rationalizing a denominator.

Example 5 Dividing imaginary numbers

Write each quotient in the form $a + bi$.

a. $\dfrac{8 - i}{2 + i}$ **b.** $\dfrac{1}{5 - 4i}$ **c.** $\dfrac{3 - 2i}{i}$

Solution

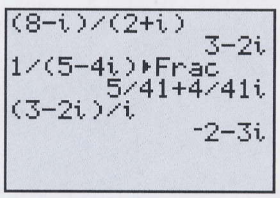
a. Multiply the numerator and denominator by $2 - i$, the conjugate of $2 + i$:

$$\frac{8 - i}{2 + i} = \frac{(8 - i)(2 - i)}{(2 + i)(2 - i)} = \frac{16 - 10i + i^2}{4 - i^2} = \frac{15 - 10i}{5} = 3 - 2i$$

Check division using multiplication: $(3 - 2i)(2 + i) = 8 - i$.

b. $$\frac{1}{5 - 4i} = \frac{1(5 + 4i)}{(5 - 4i)(5 + 4i)} = \frac{5 + 4i}{25 + 16} = \frac{5 + 4i}{41} = \frac{5}{41} + \frac{4}{41}i$$

Check: $$\left(\frac{5}{41} + \frac{4}{41}i\right)(5 - 4i) = \frac{25}{41} + \frac{20}{41}i - \frac{20}{41}i - \frac{16}{41}i^2$$
$$= \frac{25}{41} + \frac{16}{41} = 1.$$



■ Figure P.25

You can also check with a calculator that handles complex numbers, as in Fig. P.25. □

c. $$\frac{3 - 2i}{i} = \frac{(3 - 2i)(-i)}{i(-i)} = \frac{-3i + 2i^2}{-i^2} = \frac{-2 - 3i}{1} = -2 - 3i$$

Check: $(-2 - 3i)(i) = -3i^2 - 2i = 3 - 2i$.

Try This. Write $\dfrac{4}{1 + i}$ in the form $a + bi$. ■

Roots of Negative Numbers

In Examples 2(d) and 2(e), we saw that both $(3i)^2 = -9$ and $(-3i)^2 = -9$. This means that in the complex number system there are two square roots of -9, $3i$ and $-3i$. For any positive real number b, we have $\left(i\sqrt{b}\right)^2 = -b$ and $\left(-i\sqrt{b}\right)^2 = -b$. So there are two square roots of $-b$, $i\sqrt{b}$ and $-i\sqrt{b}$. We call $i\sqrt{b}$ the **principal square root** of $-b$ and make the following definition.

Definition: Square Root of a Negative Number

For any positive real number b, $\sqrt{-b} = i\sqrt{b}$.

In the real number system, $\sqrt{-2}$ and $\sqrt{-8}$ are undefined, but in the complex number system they are defined as $\sqrt{-2} = i\sqrt{2}$ and $\sqrt{-8} = i\sqrt{8}$. Even though we now have meaning for a symbol such as $\sqrt{-2}$, *all operations with complex numbers must be performed after converting to the $a + bi$ form.* If we perform operations with roots of negative numbers using properties of the real numbers, we can get contradictory results:

$$\sqrt{-2} \cdot \sqrt{-8} = \sqrt{(-2)(-8)} = \sqrt{16} = 4 \quad \text{Incorrect.}$$

$$i\sqrt{2} \cdot i\sqrt{8} = i^2 \cdot \sqrt{16} = -4 \quad \text{Correct.}$$

The product rule $\sqrt{a} \cdot \sqrt{b} = \sqrt{ab}$ is used *only* for nonnegative numbers a and b.

Example 6 Square roots of negative numbers

Write each expression in the form $a + bi$, where a and b are real numbers.

a. $\sqrt{-8} + \sqrt{-18}$ **b.** $\dfrac{-4 + \sqrt{-50}}{4}$ **c.** $\sqrt{-27}\left(\sqrt{9} - \sqrt{-2}\right)$

Solution

The first step in each case is to replace the square roots of negative numbers by expressions with i.

a. $$\sqrt{-8} + \sqrt{-18} = i\sqrt{8} + i\sqrt{18} = 2i\sqrt{2} + 3i\sqrt{2} = 5i\sqrt{2}$$

b. $$\frac{-4 + \sqrt{-50}}{4} = \frac{-4 + i\sqrt{50}}{4} = \frac{-4 + 5i\sqrt{2}}{4} = -1 + \frac{5}{4}i\sqrt{2}$$

c. $$\sqrt{-27}\left(\sqrt{9} - \sqrt{-2}\right) = 3i\sqrt{3}\left(3 - i\sqrt{2}\right) = 9i\sqrt{3} - 3i^2\sqrt{6} = 3\sqrt{6} + 9i\sqrt{3}$$

Try This. Write $\dfrac{2 - \sqrt{-12}}{2}$ in the form $a + bi$. ■

For Thought

True or False? Explain.

1. The multiplicative inverse of i is $-i$. T
2. The conjugate of i is $-i$. T
3. The set of complex numbers is a subset of the set of real numbers. F
4. $\left(\sqrt{3} - i\sqrt{2}\right)\left(\sqrt{3} + i\sqrt{2}\right) = 5$ T
5. $(2 + 5i)(2 + 5i) = 4 + 25$ F
6. $5 - \sqrt{-9} = 5 - 9i$ F
7. $(3i)^2 + 9 = 0$ T
8. $(-3i)^2 + 9 = 0$ T
9. $i^4 = 1$ T
10. $i^{18} = 1$ F

P.4 Exercises

Determine whether each complex number is real or imaginary and write it in the standard form a + bi. (Example 1)

1. $6i$ Imaginary, $0 + 6i$
2. $-3i + \sqrt{6}$ Imaginary, $\sqrt{6} - 3i$
3. $\dfrac{1+i}{3}$ Imaginary, $\dfrac{1}{3} + \dfrac{1}{3}i$
4. -72 Real, $-72 + 0i$
5. $\sqrt{7}$ Real, $\sqrt{7} + 0i$
6. $-i\sqrt{5}$ Imaginary, $0 - \sqrt{5}i$
7. $\dfrac{\pi}{2}$ Real, $\dfrac{\pi}{2} + 0i$
8. 0 Real, $0 + 0i$

Perform the indicated operations and write your answers in the form a + bi, where a and b are real numbers. (Examples 2 and 3)

9. $(3 - 3i) + (4 + 5i)$ $7 + 2i$
10. $(-3 + 2i) + (5 - 6i)$ $2 - 4i$
11. $(1 - i) - (3 + 2i)$ $-2 - 3i$
12. $(6 - 7i) - (3 - 4i)$ $3 - 3i$
13. $(1 - i\sqrt{2}) + (3 + 2i\sqrt{2})$ $4 + i\sqrt{2}$
14. $(5 + 3i\sqrt{5}) + (-4 - 5i\sqrt{5})$ $1 - 2i\sqrt{5}$
15. $\left(5 + \dfrac{1}{3}i\right) - \left(\dfrac{1}{2} - \dfrac{1}{2}i\right)$ $\dfrac{9}{2} + \dfrac{5}{6}i$
16. $\left(\dfrac{1}{2} - \dfrac{2}{3}i\right) - \left(3 - \dfrac{1}{4}i\right)$†
17. $-6i(3 - 2i)$ $-12 - 18i$
18. $-3i(5 + 2i)$ $6 - 15i$
19. $(2 - 3i)(4 + 6i)$ 26
20. $(3 - i)(5 - 2i)$ $13 - 11i$
21. $(4 - 5i)(6 + 2i)$ $34 - 22i$
22. $(3 + 7i)(2 + 5i)$ $-29 + 29i$
23. $(5 - 2i)(5 + 2i)$ 29
24. $(4 + 3i)(4 - 3i)$ 25
25. $(\sqrt{3} - i)(\sqrt{3} + i)$ 4
26. $(\sqrt{2} + i\sqrt{3})(\sqrt{2} - i\sqrt{3})$ 5
27. $(3 + 4i)^2$ $-7 + 24i$
28. $(-6 - 2i)^2$ $32 + 24i$
29. $(\sqrt{5} - 2i)^2$ $1 - 4i\sqrt{5}$
30. $(\sqrt{6} + i\sqrt{3})^2$ $3 + 6i\sqrt{2}$
31. i^{17} i
32. i^{24} 1
33. i^{98} -1
34. i^{19} $-i$
35. i^{-1} $-i$
36. i^{-2} -1
37. i^{-3} i
38. i^{-4} 1
39. i^{-13} $-i$
40. i^{-27} i
41. i^{-38} -1
42. i^{-66} -1

Find the product of the given complex number and its conjugate. (Example 4)

43. $3 - 9i$ 90
44. $4 + 3i$ 25
45. $\dfrac{1}{2} + 2i$ $17/4$
46. $\dfrac{1}{3} - i$ $10/9$
47. i 1
48. $-i\sqrt{5}$ 5
49. $3 - i\sqrt{3}$ 12
50. $\dfrac{5}{2} + i\dfrac{\sqrt{2}}{2}$ $27/4$

Write each quotient in the form a + bi. (Example 5)

51. $\dfrac{1}{2 - i}$ $\dfrac{2}{5} + \dfrac{1}{5}i$
52. $\dfrac{1}{5 + 2i}$ $\dfrac{5}{29} - \dfrac{2}{29}i$
53. $\dfrac{-3i}{1 - i}$ $\dfrac{3}{2} - \dfrac{3}{2}i$
54. $\dfrac{3i}{-2 + i}$ $\dfrac{3}{5} - \dfrac{6}{5}i$
55. $\dfrac{-3 + 3i}{i}$ $3 + 3i$
56. $\dfrac{-2 - 4i}{-i}$ $4 - 2i$
57. $\dfrac{1 - i}{3 + 2i}$ $\dfrac{1}{13} - \dfrac{5}{13}i$
58. $\dfrac{4 + 2i}{2 - 3i}$ $\dfrac{2}{13} + \dfrac{16}{13}i$
59. $\dfrac{2 - i}{3 + 5i}$ $\dfrac{1}{34} - \dfrac{13}{34}i$
60. $\dfrac{4 + 2i}{5 - 3i}$ $\dfrac{7}{17} + \dfrac{11}{17}i$

Write each expression in the form a + bi, where a and b are real numbers. (Example 6)

61. $\sqrt{-4} - \sqrt{-9}$ $-i$
62. $\sqrt{-16} + \sqrt{-25}$ $9i$
63. $\sqrt{-4} - \sqrt{16}$ $-4 + 2i$
64. $\sqrt{-3} \cdot \sqrt{-3}$ -3
65. $(\sqrt{-6})^2$ -6
66. $(\sqrt{-5})^3$ $-5i\sqrt{5}$
67. $\sqrt{-2} \cdot \sqrt{-50}$ -10
68. $\dfrac{-6 + \sqrt{-3}}{3}$ $-2 + \dfrac{\sqrt{3}}{3}i$
69. $\dfrac{-2 + \sqrt{-20}}{2}$ $-1 + i\sqrt{5}$
70. $\dfrac{9 - \sqrt{-18}}{-6}$ $-\dfrac{3}{2} + \dfrac{\sqrt{2}}{2}i$
71. $-3 + \sqrt{3^2 - 4(1)(5)}$ $-3 + i\sqrt{11}$
72. $1 - \sqrt{(-1)^2 - 4(1)(1)}$ $1 - i\sqrt{3}$
73. $\sqrt{-8}(\sqrt{-2} + \sqrt{8})$ $-4 + 8i$
74. $\sqrt{-6}(\sqrt{2} - \sqrt{-3})$ $3\sqrt{2} + 2i\sqrt{3}$

Evaluate the expression $\dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$ *for each choice of a, b, and c.* (Example 6)

75. $a = 1, b = 2, c = 5$ $-1 + 2i$
76. $a = 5, b = -4, c = 1$ $\dfrac{2}{5} + \dfrac{1}{5}i$
77. $a = 2, b = 4, c = 3$ $\dfrac{-2 + i\sqrt{2}}{2}$
78. $a = 2, b = -4, c = 5$ $\dfrac{2 + i\sqrt{6}}{2}$

Evaluate the expression $\dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$ *for each choice of a, b, and c.* (Example 6)

79. $a = 1, b = 6, c = 17$ $-3 - 2i\sqrt{2}$
80. $a = 1, b = -12, c = 84$ $6 - 4i\sqrt{3}$
81. $a = -2, b = 6, c = 6$ $\dfrac{3 + \sqrt{21}}{2}$
82. $a = 3, b = 6, c = 8$ $\dfrac{-3 - i\sqrt{15}}{3}$

Perform the indicated operations. Write the answers in the form a + bi where a and b are real numbers.

83. $(3 - 5i)(3 + 5i)$ 34
84. $(2 - 4i)(2 + 4i)$ 20

†Due to space constrictions, answers to these exercises may be found in the complete Answers beginning on page A-1 in the back of the book.

85. $(3 - 5i) + (3 + 5i)$ 6

86. $(2 - 4i) + (2 + 4i)$ 4

87. $\dfrac{3 - 5i}{3 + 5i}$ $-\dfrac{8}{17} - \dfrac{15}{17}i$

88. $\dfrac{2 - 4i}{2 + 4i}$ $-\dfrac{3}{5} - \dfrac{4}{5}i$

89. $(6 - 2i) - (7 - 3i)$ $-1 + i$

90. $(5 - 6i) - (8 - 9i)$ $-3 + 3i$

91. $i^5(i^2 - 3i)$ $3 - i$

92. $3i^7(i - 5i^3)$ 18

For Writing/Discussion

93. Explain in detail how to find i^n for any positive integer n.

94. Find a number $a + bi$ such that $a^2 + b^2$ is irrational. $\sqrt[4]{2} + i\sqrt[4]{3}$

95. Let $w = a + bi$ and $\overline{w} = a - bi$, where a and b are real numbers. Show that $w + \overline{w}$ is real and that $w - \overline{w}$ is imaginary. Write sentences (containing no mathematical symbols) stating these results.

96. Is it true that the product of a complex number and its conjugate is a real number? Explain. True

97. Prove that the reciprocal of $a + bi$, where a and b are not both zero, is $\dfrac{a}{a^2 + b^2} - \dfrac{b}{a^2 + b^2}i$.

98. *Cooperative Learning* Work in a small group to find the two square roots of 1 and two square roots of -1 in the complex number system. How many fourth roots of 1 are there in the complex number system and what are they? Explain how to find all of the fourth roots in the complex number system for any positive real number. ± 1, $\pm i$, Fourth roots of 1 are ± 1 and $\pm i$. Fourth roots of x for $x > 0$ are $\pm\sqrt[4]{x}$ and $\pm i\sqrt[4]{x}$.

99. Evaluate $i^{0!} + i^{1!} + i^{2!} + \cdots + i^{100!}$, where $n!$ (read "n factorial") is the product of the integers from 1 through n if $n \geq 1$ and $0! = 1$. $95 + 2i$

Thinking Outside the Box VI & VII

Reversing the Digits Find a four-digit integer x such that $4x$ is another four-digit integer whose digits are in the reverse order of the digits of x. 2178

Summing Reciprocals There is only one way to write 1 as a sum of the reciprocals of three different positive integers:

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{6} = 1$$

Find all possible ways to write 1 as a sum of the reciprocals of four different positive integers.
$\frac{1}{2} + \frac{1}{3} + \frac{1}{10} + \frac{1}{15}$, $\frac{1}{2} + \frac{1}{3} + \frac{1}{9} + \frac{1}{18}$, $\frac{1}{2} + \frac{1}{3} + \frac{1}{8} + \frac{1}{24}$,
$\frac{1}{2} + \frac{1}{3} + \frac{1}{7} + \frac{1}{42}$, $\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{12}$, $\frac{1}{2} + \frac{1}{4} + \frac{1}{5} + \frac{1}{20}$

P.4 Pop Quiz

1. Find the sum of $3 + 2i$ and $4 - i$. $7 + i$
2. Find the product of $4 - 3i$ and $2 + i$. $11 - 2i$.
3. Find the product of $2 - 3i$ and its conjugate. 13
4. Write $\dfrac{5}{2 - 3i}$ in the form $a + bi$. $\dfrac{10}{13} + \dfrac{15}{13}i$
5. Find i^{27}. $-i$
6. What are the two square roots of -16? $\pm 4i$